A computer-assisted spectral gap for the spherical fermionic V_1 pseudopotential

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Abstract

We give a computer-assisted proof of a uniform gap above the zero modes of the fully spin-polarized fermionic Haldane V_1 pseudopotential on a sphere. With Q magnetic flux quanta and each forbidden two-particle channel normalized to energy one, we prove that there is Q_0 such that

$$H_{n,Q}^2 \geq g_* H_{n,Q}, \qquad g_* = \frac{15736921879}{10^{12}},$$

for every $Q \geq Q_0$ and every particle-number sector. In particular, the Laughlin sequence $Q = 3(N-1)$ has a positive thermodynamic spectral gap. The proof retains a controlled portion of the positive four-particle remainder in the square of the Hamiltonian. A fermionic contraction estimate bounds the discarded three-particle angular-momentum channels uniformly in particle number. The remaining inequality follows by rotationally averaging a positive operator on the complete Fock space of fifteen orbitals. Its passage from planar coefficients to finite spherical coefficients preserves the reference kernel exactly, so that finite-dimensional continuity can be used without assuming the desired many-particle gap. The local inequality is verified in all 32 768 Fock states by exact rational assembly, exact kernel identities, and outward-rounded interval arithmetic. All coefficients, verification algorithms, and input-data hashes are specified. The numerical lower bound is eventual: the argument does not provide an explicit value of Q_0. No assertion about a Coulomb interaction, a torus, or perturbative stability is made.

Keywords. Fractional quantum Hall effect; Laughlin state; spectral gap; Haldane pseudopotential; fermionic Fock space; computer-assisted proof.

Verification status. This manuscript presents a new proof claim with a reproducible finite certificate. It does not report independent referee validation or proof-assistant formalization. The exact computational statement and its trusted arithmetic operations are isolated in Section 8.

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1 Introduction and main result

Laughlin's wavefunction identifies a strongly correlated state at fractional filling, and the spherical formulation of Haldane gives a boundary-free setting in which to study its parent interaction [3, 1]. The short-range parent-Hamiltonian perspective has a long history [10]; a rigorous derivation of pseudopotentials from suitable short-range interactions is available in [9]. An exact ground-state formula, however, does not control the first positive eigenvalue uniformly in system size. This is the distinction at the center of the pseudopotential gap problem described in [8, Appendix A].

We study a single, precisely normalized Hamiltonian. Let V_j be the irreducible spin-j representation of $SU(2)$ and let $Q \in \mathbb{N}$ be the number of magnetic flux quanta, so that $j = Q/2$. The one-particle lowest Landau level has dimension $Q + 1$. On the n-fermion Hilbert space

$$\mathcal{H}_{n,Q} = \bigwedge^{n} V_{Q/2}, \qquad 0 \le n \le Q + 1, \tag{1.1}$$

let $P^{(1)}_{ik}$ be the pair projector of relative angular momentum one, equivalently total pair spin $J = Q - 1$. The Hamiltonian is

$$H_{n,Q} = \sum_{1 \le i < k \le n} P^{(1)}_{ik}. \tag{1.2}$$

The superscript (1) denotes relative angular momentum, not a one-body operator. Its coefficient is one for every n and Q; no division by the particle number is performed.

The result will be proved first as an inequality on the entire finite Fock space

$$\mathfrak{H}_Q = \bigoplus_{n=0}^{Q+1} \mathcal{H}_{n,Q}. \tag{1.3}$$

When an equation contains H_Q without a particle-number subscript, it denotes this direct sum. Operators preserve particle number throughout.

Theorem 1.1 (Uniform separation of the positive spectrum)**.** *There exists an integer Q_0 such that, for all integers $Q \ge Q_0$,*

$$H_Q^2 \ge g_* H_Q, \qquad g_* = \frac{15736921879}{10^{12}} = 0.015736921879. \tag{1.4}$$

Consequently,

$$\operatorname{spec}(H_{n,Q}) \subset \{0\} \cup [g_*, \infty) \tag{1.5}$$

for every $0 \le n \le Q + 1$.

Corollary 1.2 (Spherical Laughlin gap)**.** *For $N \ge 2$, set $Q_N = 3(N - 1)$ and let Δ_N be the smallest positive eigenvalue of H_{N,Q_N}. Then*

$$\liminf_{N\to\infty} \Delta_N \ge g_*, \qquad \inf_{N\ge 2} \Delta_N > 0. \tag{1.6}$$

In fact, $\Delta_N \ge g_*$ for every sufficiently large N.

The unique zero mode at Q_N is the spherical Laughlin state; its elementary algebraic proof is included in Proposition 2.2. The finitely many values of N not covered by Theorem 1.1 have positive finite-dimensional gaps. Their minimum with g_* proves the second assertion of (1.6). Neither an explicit Q_0 nor the numerical bound $\Delta_N \ge g_*$ for every small N is asserted.

Remark 1.3 (Meaning of the all-sector statement)**.** For a sector with no zero modes, (1.4) bounds its absolute energy above zero. It need not bound the spacing between its two lowest positive eigenvalues. On the Laughlin sequence, zero is the ground-state energy, so the separation from zero is exactly the excitation gap. Vacuum and one-particle sectors are identically zero and cause no difficulty.

1.1 Geometry and scope

At fixed magnetic length ℓ_B, the sphere has radius R_Q determined by

$$R_Q^2 = \frac{Q\ell_B^2}{2}. \tag{1.7}$$

Thus the sequence in Corollary 1.2 grows in both spatial directions. No thin-cylinder limit or physical boundary is introduced. All fermions are fully spin polarized; physical spin is not an additional degree of freedom.

The theorem concerns the parent interaction (1.2). It does not identify its gap with the gap of a projected Coulomb Hamiltonian. Nor does it establish a gap on another compact surface, stability under perturbations, or a construction of an infinite-volume ground-state representation. The thermodynamic statement here is the uniform finite-sphere spectral inequality (1.6).

1.2 Relation to previous methods

Rigorous gap and incompressibility results for truncated fractional quantum Hall Hamiltonians are known in thin-cylinder and thin-torus regimes [5, 11]. The final Hamiltonian in the present argument is not truncated. Only an auxiliary comparison operator uses finitely many angular channels, and the omitted negative terms are bounded in operator form.

The square inequality $H^2 \geq gH$ is a standard route to a frustration-free spectral gap. Finite-size criteria initiated by Knabe [2], and semidefinite hierarchies of spectral-gap certificates [7], exemplify this strategy for spin systems. Here the finite object is a complete fermionic orbital Fock space, and the averaging group is $SU(2)$ rather than a lattice translation group. Pairing representations of quantum Hall interactions are also useful background [6].

We use the three-fermion single-j, single-J pairing spectrum of Zhao and Arima [12], changing the interaction sign and retaining the fermionic exchange convention for both integral and half-integral orbital j. The three-particle solution is not claimed as new. Particle-number recursions give another route toward gaps [4]; no closure of such a recursion is assumed here.

1.3 Proof architecture and the role of the computer

The exact identity

$$H_Q^2 = H_Q + \sum_m w_m(Q) K_{m,Q} + \mathcal{D}_Q \tag{1.8}$$

separates two-, three-, and four-body contributions. The operators $K_{m,Q}$ are positive lifts of active three-particle channel projectors. Some coefficients $w_m(Q)$ are negative. The disjoint-pair term $\mathcal{D}_Q$ is positive.

A particle-number-independent estimate bounds the negative channels above a cutoff M by $\epsilon_M H_Q$. We retain $M = 14$. A positive local operator S_Q is then chosen so that its rotational average is

$$(2Q-1)\mathcal{T}(S_Q) = sH_Q + \sum_{m\leq 14} w_m(Q) K_{m,Q} + \text{a term bounded by } \mathcal{D}_Q. \tag{1.9}$$

It follows that $H_Q^2 \geq (1 - s - \epsilon_{14}) H_Q$.

The local operator is first specified in a fixed fifteen-orbital planar coefficient space. The sole computer-assisted inequality is

$$S \geq 10^{-4} h_{14} \quad \text{on } \mathfrak{F}(\mathbb{C}^{15}), \tag{1.10}$$

where h_{14} is a fixed positive reference operator. Exact kernel alignment proves that (1.10) persists as positivity of S_Q on large finite spheres. No finite-size extrapolation and no estimate of a thermodynamic reference gap are used.

Retaining part of $\mathcal{D}_Q$ is essential to the certificate presented here. It controls local sectors in which several pair constraints are violated at once. A further improvement comes from the fact that the two highest symmetric pair-pair angular channels cannot occur for four identical fermions. That exclusion makes the four-body term less expensive under rotational averaging.

2 Fock-space conventions and the exact square identity

2.1 Creation, annihilation, and lifting

Choose an orthonormal orbital basis e_a, $0 \le a \le Q$, of $V_{Q/2}$ with magnetic quantum number $j - a$. Creation and annihilation operators satisfy

$$\{c_a, c_b^\dagger\} = \delta_{ab}\mathbf{1}, \qquad \{c_a, c_b\} = \{c_a^\dagger, c_b^\dagger\} = 0. \tag{2.1}$$

For an increasing set $I = \{a_1 < \cdots < a_k\}$ write

$$|I\rangle = c_{a_1}^\dagger \cdots c_{a_k}^\dagger |0\rangle, \qquad c_I = c_{a_k} \cdots c_{a_1}. \tag{2.2}$$

If $v = \sum_{|I|=k} v_I |I\rangle \in \bigwedge^k V_j$, its annihilator is

$$c(v) = \sum_{|I|=k} \overline{v_I}\, c_I. \tag{2.3}$$

In particular, $c(v)^\dagger|0\rangle = v$.

For a k-particle operator A define its normally ordered lift by

$$\mathcal{L}_k(A) = \sum_{|I|=|J|=k} \langle I, AJ\rangle\, c_I^\dagger c_J. \tag{2.4}$$

Here $\langle I, AJ\rangle$ means $\langle I|A|J\rangle$. On the k-particle sector the lift is exactly A, and on the n-particle sector it is the sum of A over all particle subsets of size k. There is no division by $\binom{n}{k}$.

Lemma 2.1 (Properties of the lift)**.** *The map $\mathcal{L}_k$ is linear, covariant under one-particle unitaries, and positive: $A \ge 0$ implies $\mathcal{L}_k(A) \ge 0$. Two homogeneous normally ordered k-body operators that agree on the k-particle sector agree on every Fock sector.*

Proof. If $A = \sum_\nu a_\nu |v_\nu\rangle\langle v_\nu|$ with $a_\nu \ge 0$, then $\mathcal{L}_k(A) = \sum_\nu a_\nu c(v_\nu)^\dagger c(v_\nu) \ge 0$. The creation-polynomial definition gives covariance. Finally, all coefficients of a homogeneous normally ordered k-body operator are its matrix entries between k-particle occupation states. Those coefficients determine the operator on Fock space. □

2.2 Normalized spherical pair operators

The normalized forbidden pair vectors are annihilated by

$$B_t^{(Q)} = \sum_{\substack{0\le a<b\le Q \\ a+b=t}} (b-a)\sqrt{\frac{\binom{Q}{a}\binom{Q}{b}}{Q\binom{2Q-2}{t-1}}}\, c_b c_a, \qquad 1 \le t \le 2Q-1. \tag{2.5}$$

Their magnetic quantum number is $Q - t$; their spin is $J = Q - 1$. The vector at $t = 1$ is $e_0 \wedge e_1$. Applying the total-spin lowering operator produces (2.5) with positive real coefficients in the displayed convention.

For completeness, the normalization identity is

$$\sum_{\substack{a<b\\ a+b=t}} (b-a)^2 \binom{Q}{a}\binom{Q}{b} = Q\binom{2Q-2}{t-1}. \tag{2.6}$$

It follows by summing over ordered a, b, using symmetry, and evaluating the first two moments of $a-b$ under Vandermonde's convolution. Equivalently, it follows from the normalized $SU(2)$ lowering factors starting from $e_0 \wedge e_1$. Distinct t have different magnetic quantum numbers, so the pair vectors are orthogonal.

Consequently,

$$H_Q = \sum_{t=1}^{2Q-1} (B_t^{(Q)})^\dagger B_t^{(Q)} = \mathcal{L}_2(P^{(1)}). \tag{2.7}$$

Every pair coefficient in the proof uses this convention. In particular, the two-particle Hamiltonian is a projector with nonzero eigenvalue one.

2.3 The Laughlin zero mode

Proposition 2.2 (Ground space at Laughlin flux)**.** *For $N \geq 2$ and $Q = 3(N-1)$,*

$$\ker H_{N,Q} = \operatorname{span}\left\{ \prod_{i<k} (u_i v_k - u_k v_i)^3 \right\}. \tag{2.8}$$

In particular, its zero eigenspace is one dimensional.

Proof. A lowest-Landau-level wavefunction on the sphere is represented by a polynomial homogeneous of degree Q in each spinor (u_i, v_i). Fermionic exchange permits only odd pair relative angular momenta. The two-particle spin-$Q-r$ summand carries the factor $(u_i v_k - u_k v_i)^r$. Thus exclusion of the $r = 1$ component is equivalent, on antisymmetric polynomials, to divisibility by the cube of this determinant.

Positivity of the pair-projector sum implies that a zero mode is annihilated by each pair projector. It is therefore divisible by the cube of every pair determinant. These determinants are distinct irreducible factors in the polynomial ring; hence their product divides the wavefunction. That product has degree $3(N-1) = Q$ in every spinor. Its quotient has degree zero in every variable and is constant. Conversely, the product has no forbidden pair component and is a nonzero zero mode. □

The proof identifies the kernel but supplies no lower bound on positive eigenvalues. At fixed N such a bound exists by finite dimensionality. Uniformity is the content of the rest of the paper.

2.4 The exact two-, three-, and four-body decomposition

For a labelled particle triple T, let h_T be the sum of its three pair projectors. Define

$$\mathcal{D}_Q = 2 \sum_{\substack{\{e,f\} \text{ unordered}\\ e\cap f=\varnothing}} P_e^{(1)} P_f^{(1)}. \tag{2.9}$$

Projectors on disjoint labelled particle pairs commute, so every summand is positive.

Proposition 2.3 (Square identity)**.** *On every particle-number sector,*

$$H_Q^2 = H_Q + \sum_{|T|=3} (h_T^2 - h_T) + \mathcal{D}_Q, \tag{2.10}$$

$$\mathcal{D}_Q = \sum_{s,t=1}^{2Q-1} (B_s^{(Q)} B_t^{(Q)})^\dagger B_s^{(Q)} B_t^{(Q)}. \tag{2.11}$$

The sum in (2.11) *is ordered.*

Proof. Expand the square of the sum of pair projectors. A diagonal term is $P_e^2 = P_e$. Two different pairs either share one particle or are disjoint. The shared-particle terms occur in exactly one triple; together they are $h_T^2 - h_T$. The disjoint terms give (2.9).

To obtain (2.11), normal-order (2.7) squared. The term with no contraction between the two pair factors is precisely the right-hand side of (2.11). It is a homogeneous four-body operator. Terms with one or two contractions have body order three or two and give the other terms in (2.10). Equivalently, on four particles the norm square of the ordered pair-removal map counts the two orders for each disjoint pair partition, giving (2.9). Lemma 2.1 extends this equality to every sector. □

As a normalization test, when the pair projector is the identity on $\bigwedge^2 V_j$, the four-body term is $6\binom{n}{4}\mathbf{1}$, not $3\binom{n}{4}\mathbf{1}$. This agrees both with ordered pair removal and with the factor two in (2.9).

3 Three-particle channels and a uniform tail estimate

3.1 The pair–single wedge map

Let $W_Q \cong V_J \subset \bigwedge^2 V_j$ be the forbidden pair representation, where $J = Q - 1$ and $j = Q/2$. Define

$$T_Q : W_Q \otimes V_j \longrightarrow \bigwedge^3 V_j, \qquad T_Q(w \otimes v) = w \wedge v. \tag{3.1}$$

This map uses normalized exterior products, not normalized projections from the tensor product. Its adjoint is pair removal. Therefore

$$H_{3,Q} = T_Q T_Q^\dagger. \tag{3.2}$$

The source representation is multiplicity free:

$$W_Q \otimes V_j = \bigoplus_{m=1}^{Q+1} V_{L_m}, \qquad L_m = \frac{3Q}{2} - m. \tag{3.3}$$

By Schur's lemma, $T_Q^\dagger T_Q$ acts as a scalar on each source summand. If the scalar is positive, T_Q maps that summand to one active copy of spin L_m in the three-particle space. Other copies, if present, are zero modes of $H_{3,Q}$.

Proposition 3.1 (Three-fermion spectrum)**.** *For $Q \geq 6$, the nonzero multiplets of $H_{3,Q}$ are indexed by*

$$\mathcal{I}_Q = \{3\} \cup \{5, 6, \ldots, Q+1\}. \tag{3.4}$$

There is one active copy for each $m \in \mathcal{I}_Q$, with eigenvalue

$$\lambda_m(Q) = 1 - (-1)^m t_m(Q), \tag{3.5}$$

$$t_m(Q) = B_m(Q) \frac{(Q-1)!(2Q-m-1)!}{(Q-m+1)!(2Q-2)!}, \tag{3.6}$$

$$B_m(Q) = (3m-4)Q - m(m-1). \tag{3.7}$$

The source multiplets $m = 1, 2, 4$ are annihilated by T_Q.

Proof. Identify $W_Q \otimes V_j$ with the corresponding subspace of tensors antisymmetric in the first two particle labels. If A_3 is the three-particle antisymmetrizer and P_W the source projection, then

$$T_Q^\dagger T_Q = 3P_W A_3 P_W = \mathbf{1} - 2P_W P_{23} P_W. \tag{3.8}$$

The last equality follows by expanding the six permutations in A_3 and using $P_{12} = -\mathbf{1}$ on the source.

The recoupling matrix element gives

$$\lambda_m(Q) = 1 + (-1)^{Q+1} 2(2Q-1) \begin{Bmatrix} Q-1 & Q/2 & 3Q/2-m \\ Q-1 & Q/2 & Q/2 \end{Bmatrix}. \tag{3.9}$$

This is the single-j pairing calculation of [12], with the positive interaction convention used here. The factor $(-1)^{Q+1}$ retains exterior statistics even when j is integral.

The Racah factorial formula reduces the symbol in (3.9) to

$$(-1)^{3Q-m-1} \frac{(Q-1)!(2Q-m-1)!}{(2Q)!(Q-m+1)!} \Big[(Q-m+1)^2 - (m-1)(Q-1)(3Q-m+1) \Big]. \tag{3.10}$$

For $3 \le m \le Q$, the two summation indices are $3Q-m-1$ and $3Q-m$. At $m = Q+1$, the first summand is absent and its simplified factor $(Q-m+1)^2$ vanishes. The bracket in (3.10) is $-QB_m(Q)$; substitution into (3.9) yields (3.5)–(3.6). The endpoint cases $m = 1, 2$ can be checked directly and give zero; $t_4(Q) = 1$ gives zero at $m = 4$.

For odd $m \ge 3$, $B_m(Q) > 0$, so $\lambda_m(Q) > 1$. For even $m \ge 6$, Lemma 3.4 below gives $0 < t_m(Q) \le (3m-4)/2^{m-1} < 1$. Thus every listed active scalar is positive. The source decomposition and (3.2) show that there are no other nonzero eigenvalues. □

Write $\Pi_{m,Q}$ for the projector onto the active three-particle copy and let

$$K_{m,Q} = \mathcal{L}_3(\Pi_{m,Q}), \qquad w_m(Q) = \lambda_m(Q)(\lambda_m(Q) - 1). \tag{3.11}$$

The spectral resolution on three particles, followed by Lemma 2.1, turns Proposition 2.3 into

$$H_Q^2 = H_Q + \sum_{m \in \mathcal{I}_Q} w_m(Q) K_{m,Q} + \mathcal{D}_Q. \tag{3.12}$$

The coefficients are positive for odd m and negative for even $m \ge 6$.

For each fixed m the planar limits are

$$\tau_m = \frac{3m-4}{2^{m-1}}, \qquad \lambda_m^\infty = 1 - (-1)^m \tau_m, \qquad w_m^\infty = \lambda_m^\infty(\lambda_m^\infty - 1). \tag{3.13}$$

For example, $\lambda_3^\infty = 9/4$, $\lambda_5^\infty = 27/16$, and $\lambda_6^\infty = 9/16$. These three-particle energies are not thermodynamic gap estimates.

3.2 A particle-number-independent channel inequality

The key bound uses the norm of a fermionic annihilation operator rather than a count of possible spectator particles.

Lemma 3.2 (Fermionic contraction estimate)**.** *Let $U \cong V_J$ be an irreducible auxiliary representation and $V \cong V_j$ a one-particle representation. Let $\eta \in U \otimes V$ be a normalized highest-weight vector of an irreducible spin-L summand, with Schmidt rank at most r. If $\Xi \in U \otimes \mathfrak{F}(V)$, contraction of its auxiliary component and removal of one fermion, followed by projection onto this spin-L summand, obeys*

$$\|C_L \Xi\|^2 \le r \frac{2L+1}{2J+1} \|\Xi\|^2. \tag{3.14}$$

Proof. Write a Schmidt decomposition

$$\eta = \sum_{k=1}^{r'} \alpha_k f_k \otimes g_k, \qquad r' \le r, \qquad \sum_k |\alpha_k|^2 = 1, \tag{3.15}$$

with orthonormal f_k and g_k. Put $\Xi_k = (\langle f_k| \otimes \mathbf{1})\Xi$. The contraction associated with η is

$$C_\eta \Xi = \sum_k \overline{\alpha_k}\, c(g_k)\Xi_k. \tag{3.16}$$

The canonical anticommutation relations give $\|c(g_k)\| \le 1$. Cauchy–Schwarz for a sum of r' vectors therefore implies

$$\|C_\eta \Xi\|^2 \le r' \sum_k |\alpha_k|^2 \|\Xi_k\|^2. \tag{3.17}$$

The projector onto spin L has the coherent-state resolution

$$\Pi_L = (2L+1) \int |\eta_R\rangle\langle\eta_R|\, dR, \tag{3.18}$$

where dR is normalized Haar measure and $\eta_R = (U_J(R) \otimes U_j(R))\eta$. Apply (3.17) at every rotation. The pair-space density matrix $\sum_k |\alpha_k|^2 |f_k\rangle\langle f_k|$ has trace one. Its rotational average is $\mathbf{1}_U/(2J+1)$. Integrating gives (3.14). □

Proposition 3.3 (Uniform channel form bound)**.** *For $m \in \mathcal{I}_Q$ and every particle-number sector,*

$$\lambda_m(Q) K_{m,Q} \le \frac{m(3Q-2m+1)}{2Q-1} H_Q \le \frac{3m}{2} H_Q. \tag{3.19}$$

Proof. Use the pair-removal map

$$\mathbf{B}_Q \psi = \sum_{t=1}^{2Q-1} |J, Q-t\rangle \otimes B_t^{(Q)} \psi, \qquad \|\mathbf{B}_Q \psi\|^2 = \langle \psi, H_Q \psi\rangle. \tag{3.20}$$

Removing one further fermion and projecting the pair–single auxiliary space onto spin L_m gives a homogeneous three-body quadratic form. On three particles it equals $\lambda_m(Q)\Pi_{m,Q}$, by the wedge-map Gram calculation. Hence on every sector it is $\lambda_m(Q) K_{m,Q}$.

A highest-weight vector of spin $L_m = J + j - (m-1)$ has lowering deficit $m-1$. It is a sum of at most m terms of the form $|J, J-p\rangle \otimes |j, j-(m-1-p)\rangle$. Its Schmidt rank is at most m. Lemma 3.2 gives the first inequality. Since $m \ge 3$, $(3Q-2m+1)/(2Q-1) \le 3/2$, proving the second. □

This estimate applies to arbitrary vectors, not only to states of low density or to vectors near a Laughlin state. It is therefore suitable for a uniform operator comparison.

3.3 Uniform recoupling estimate and the tail

Lemma 3.4. *For $Q \ge 6$ and $4 \le m \le Q+1$,*

$$0 < t_m(Q) \le \tau_m = \frac{3m-4}{2^{m-1}}. \tag{3.21}$$

Proof. The initial value is $t_4(Q) = \tau_4 = 1$. For $4 \le m \le Q$,

$$\frac{t_{m+1}(Q)}{t_m(Q)} = \frac{B_{m+1}(Q)}{B_m(Q)} \frac{Q-m+1}{2Q-m-1}. \tag{3.22}$$

All denominators and the $B_m(Q)$ in this range are positive. Comparison with $\tau_{m+1}/\tau_m = (3m-1)/(2(3m-4))$ reduces to

$$\begin{aligned} &(3m-1)B_m(Q)(2Q-m-1) - 2(3m-4)B_{m+1}(Q)(Q-m+1) \\ &\quad = (m-1)\Big[(9m^2-27m+12)(Q-m+1) + 2(m-3)(m-2)(3m-1)\Big] > 0. \end{aligned} \tag{3.23}$$

Induction proves the assertion, including the endpoint $m = Q+1$. □

Proposition 3.5 (Uniform finite-channel reduction)**.** *Let $M \geq 6$ be even. For every $Q \geq 6$,*

$$H_Q^2 \geq H_Q + \sum_{\substack{m \in I_Q \\ m \leq M}} w_m(Q) K_{m,Q} + \mathcal{D}_Q - \epsilon_M H_Q, \tag{3.24}$$

where

$$\epsilon_M = \frac{3}{2} \sum_{\substack{m > M \\ m \text{ even}}} m\tau_m = \frac{16(3a^2+1)}{4^a}, \qquad a = M/2 + 1. \tag{3.25}$$

Proof. Drop the omitted odd-m terms in (3.12), which are positive. For an even omitted m, use $1 - \lambda_m(Q) = t_m(Q)$ and Proposition 3.3:

$$-w_m(Q) K_{m,Q} = t_m(Q)\lambda_m(Q) K_{m,Q} \leq \frac{3}{2} m\tau_m H_Q. \tag{3.26}$$

Extending the finite negative sum to all even $m > M$ increases its upper bound. Summing the resulting geometric series and its first two derivatives gives (3.25). □

At the cutoff used below,

$$\epsilon_{14} = \frac{193}{4096} = 0.047119140625. \tag{3.27}$$

For comparison, $\epsilon_{12} = 37/256$, $\epsilon_{16} = 61/4096$, and $\epsilon_{24} = 127/1048576$. These are rigorous bounds for discarded angular channels, not estimates obtained by fitting finite-size gaps.

4 Rotational averaging and the four-body budget

4.1 Averaging finite-body operators

Let $\Gamma_Q(R)$ be the second quantization of the spin-$Q/2$ rotation. Define

$$\mathcal{T}_Q(A) = \int_{SU(2)} \Gamma_Q(R) A \Gamma_Q(R)^\dagger \, dR. \tag{4.1}$$

This map is positive and commutes with the lifting maps in Lemma 2.1. If v is a normalized vector in one irreducible copy of spin L, Schur's lemma gives

$$\int |v_R\rangle\langle v_R| \, dR = \frac{\Pi_L}{2L+1}. \tag{4.2}$$

In particular,

$$\mathcal{T}_Q\big((B_t^{(Q)})^\dagger B_t^{(Q)}\big) = \frac{H_Q}{2Q-1}, \tag{4.3}$$

$$\mathcal{T}_Q\big((C_{m,t}^{(Q)})^\dagger C_{m,t}^{(Q)}\big) = \frac{K_{m,Q}}{3Q-2m+1}. \tag{4.4}$$

Here $C_{m,t}^{(Q)}$ annihilates the normalized active three-particle state of spin L_m and total orbital degree t. Its magnetic quantum number is $3Q/2 - t$. The off-diagonal averages with different m vanish. The representations have different spins, even when the two vectors have the same magnetic quantum number.

4.2 A local window in pair space

Fix an integer $D \geq 6$. In the pair representation W_Q let P_D project onto the first D pair modes, $t = 1, \ldots, D$. Their lowering deficits are $0, \ldots, D-1$. Define

$$R_D^{(Q)} = \sum_{s,t=1}^{D} (B_s^{(Q)} B_t^{(Q)})^\dagger B_s^{(Q)} B_t^{(Q)}. \tag{4.5}$$

The four-particle total orbital degree of these terms can be as large as $2D$. Restricting it further to degree at most D would change the certificate.

The pair-pair removal map

$$\mathbf{Q}_Q \psi = \sum_{s,t=1}^{2Q-1} |J, Q-s\rangle \otimes |J, Q-t\rangle \otimes B_s^{(Q)} B_t^{(Q)} \psi \tag{4.6}$$

satisfies

$$\mathbf{Q}_Q^\dagger \mathbf{Q}_Q = \mathcal{D}_Q, \qquad R_D^{(Q)} = \mathbf{Q}_Q^\dagger (P_D \otimes P_D \otimes \mathbf{1}) \mathbf{Q}_Q. \tag{4.7}$$

Since pair-annihilation polynomials have even degree, they commute. The auxiliary pair tensor is therefore symmetric.

4.3 The four-fermion exclusion

The symmetric tensor product of two spin-J representations decomposes, with multiplicity one, into spins

$$L_r = 2J - r = 2Q - 2 - r, \qquad r \text{ even}. \tag{4.8}$$

Let Π_r^{pp} denote these source projectors.

Lemma 4.1 (Pauli-forbidden pair-pair channels)**.** *The components $r = 0, 2$ of $\mathbf{Q}_Q$ vanish identically in every particle-number sector.*

Proof. The maximum total spin of four identical fermions in V_j is

$$j + (j-1) + (j-2) + (j-3) = 4j - 6 = 2Q - 6. \tag{4.9}$$

For $r = 0, 2$, the source spins $2Q-2$ and $2Q-4$ exceed this maximum. The four-particle wedge-creation intertwiners from these source representations to $\bigwedge^4 V_j$ must vanish. Their adjoints are homogeneous four-annihilation polynomials. A creation polynomial that vanishes on the vacuum has all of its occupation coefficients zero; its adjoint therefore vanishes on every Fock sector. □

The lemma is not a restriction on the total spin of an arbitrary n-particle vector. It is a representation-theoretic statement about the four-particle creation map and hence an exact operator identity.

4.4 An exact finite-flux budget

Set

$$k_r(Q, D) = \mathrm{Tr}\left[(P_D \otimes P_D)\Pi_r^{\mathrm{pp}}\right]. \tag{4.10}$$

The local tensor has total lowering deficit at most $2D-2$, so $k_r(Q, D) = 0$ for $r > 2D-2$.

Proposition 4.2 (Budget for the averaged four-body term)**.** *For sufficiently large Q, define*

$$\alpha_Q(D) = \min_{\substack{4\le r\le 2D-2\\ r\ \text{even},\ k_r(Q,D)>0}} \frac{4Q-3-2r}{(2Q-1)k_r(Q,D)}. \tag{4.11}$$

Then

$$(2Q-1)\alpha_Q(D)\mathcal{T}_Q(R_D^{(Q)}) \le \mathcal{D}_Q. \tag{4.12}$$

Proof. The map $\mathbf{Q}_Q$ is equivariant. Average the auxiliary operator $P_D \otimes P_D$ in (4.7). On each spin-L_r representation its average is $k_r(Q,D)/(2L_r+1)$ times the identity. The odd-r source components do not occur in the symmetric tensor, and Lemma 4.1 removes $r = 0, 2$. Hence

$$\mathcal{T}_Q(R_D^{(Q)}) = \mathbf{Q}_Q^\dagger \left[\sum_{\substack{4\le r\le 2D-2\\ r\ \text{even}}} \frac{k_r(Q,D)}{4Q-3-2r} \Pi_r^{\mathrm{pp}} \otimes \mathbf{1} \right] \mathbf{Q}_Q. \tag{4.13}$$

The choice (4.11) makes every nonzero coefficient in brackets, after multiplication by $(2Q-1)\alpha_Q(D)$, at most one. Use the first identity in (4.7). □

All bounds in Proposition 4.2 are exact at finite Q. Only the convergence of its scalar budget will be used to deform the finite certificate.

4.5 The oscillator limit of the budget

For fixed D, each $k_r(Q,D)$ converges to an explicitly computable rational number. Use the Bargmann norm in two variables, in which $x^a y^b$ has squared norm $a!b!$. The limiting normalized source vector of relative degree r and center-of-mass degree d is

$$\frac{(x-y)^r (x+y)^d}{\sqrt{2^{r+d} r! d!}}. \tag{4.14}$$

Writing $\ell = r + d$, its coefficient in the normalized monomial of degree $(a, \ell - a)$ is

$$f_{r,d}(a) = \left[\sum_{k=\max(0,a-d)}^{\min(r,a)} (-1)^{r-k} \binom{r}{k} \binom{d}{a-k} \right] \sqrt{\frac{a!(\ell-a)!}{2^\ell r! d!}}. \tag{4.15}$$

Therefore

$$k_r(D) := \lim_{Q\to\infty} k_r(Q,D) = \sum_{\ell=r}^{2D-2} \sum_{a=\max(0,\ell-D+1)}^{\min(D-1,\ell)} f_{r,\ell-r}(a)^2. \tag{4.16}$$

Every squared coefficient and hence every $k_r(D)$ is rational.

To justify the limit without an asymptotic assumption, start with the highest-weight polynomial

$$(u_1 v_2 - v_1 u_2)^r u_1^{2J-r} u_2^{2J-r} \tag{4.17}$$

in $V_J \otimes V_J$. In normalized spin bases its coefficients involve finitely many binomial factors $\binom{2J}{a}$, with $a \le 2D-2$. After normalization they tend to the coefficients of $(x-y)^r$. Normalized total-spin lowering tends to center-of-mass raising by $x+y$. Only finitely many descendants contribute to (4.16), so coefficientwise convergence suffices. The exact local-to-oscillator construction is also described in Appendix A.

The allowed limiting budget is

$$\alpha(D) = \frac{2}{\max_{4\le r\le 2D-2,\ r\ \text{even}} k_r(D)}, \qquad \alpha_Q(D) \longrightarrow \alpha(D). \tag{4.18}$$

For $D = 14$, all allowed $k_r(14)$ are positive, and their values are listed in Table 1. The maximum occurs at $r = 4$:

$$k_4(14) = \frac{59353161}{4194304}, \qquad \alpha(14) = \frac{8388608}{59353161}. \tag{4.19}$$

The exact sum including the two forbidden auxiliary channels satisfies

$$\sum_{r=0,2,\dots,26} k_r(14) = \dim \mathrm{Sym}^2(\mathbb{C}^{14}) = 105. \tag{4.20}$$

This trace identity is an independent normalization check; the forbidden channels are not charged against $\mathcal{D}_Q$.

Table 1: Exact limiting traces $k_r(14)$ for the allowed symmetric pair-pair channels. The largest value is at $r = 4$.

r	$k_r(14)$	r	$k_r(14)$
4	$\frac{59353161}{4194304}$	16	$\frac{30530689}{8388608}$
6	$\frac{48418965}{4194304}$	18	$\frac{22242409}{8388608}$
8	$\frac{79055165}{8388608}$	20	$\frac{7590177}{4194304}$
10	$\frac{64080205}{8388608}$	22	$\frac{4650877}{4194304}$
12	$\frac{12810595}{2097152}$	24	$\frac{4628267}{8388608}$
14	$\frac{10035823}{2097152}$	26	$\frac{1300075}{8388608}$

5 A finite-certificate criterion

5.1 The fixed local Fock space

Let $D \geq M \geq 6$, with M even, and define

$$\mathfrak{F}_D^{\mathrm{loc}} = \mathfrak{F}\big(\operatorname{span}\{e_0, \dots, e_D\}\big) \cong \mathfrak{F}(\mathbb{C}^{D+1}). \tag{5.1}$$

This entire space is used, not only particle numbers up to three or four. Define the planar pair annihilators

$$b_t = \sum_{\substack{0 \leq a < b \leq D \\ a+b=t}} (b-a)\sqrt{\frac{(t-1)!}{2^{t-1}a!b!}}\, c_b c_a, \qquad 1 \leq t \leq D, \tag{5.2}$$

and the reference operator

$$h_D = \sum_{t=1}^{D} b_t^\dagger b_t. \tag{5.3}$$

The limit in (5.2) is obtained from (2.5) at fixed t. The reference operator excludes all pair indices above D. It is not the full interaction restricted to a disk, and its gap is not being substituted for a physical thermodynamic gap.

For $m \in \mathcal{I}_M := \{3\} \cup \{5, \dots, M\}$ and $m \leq t \leq D$, let $C_{m,t}$ be the annihilator of the normalized planar active triple vector of total orbital degree t. Section 7.1 gives an exact integer-polynomial definition, and Appendix A proves convergence of the normalized spherical vectors to these vectors.

Choose real $a_1, \dots, a_D$ and, for each t, a real symmetric matrix $W^{(t)}$ indexed by the retained active $m \le t$. Set

$$S_{2,3}(a, W) = \sum_{t=1}^{D} a_t b_t^\dagger b_t + \sum_{t=3}^{D} \sum_{\substack{m,m' \in \mathcal{I}_M \\ m,m' \le t}} W_{mm'}^{(t)} C_{m,t}^\dagger C_{m',t}, \tag{5.4}$$

$$R_D = \sum_{s,t=1}^{D} (b_s b_t)^\dagger b_s b_t, \tag{5.5}$$

$$S(a, W, \alpha) = S_{2,3}(a, W) + \alpha R_D. \tag{5.6}$$

The full off-diagonal sum in (5.4) includes both orders when $m \neq m'$. All operators preserve both local particle number and total orbital degree.

Theorem 5.1 (Finite certificate implies an eventual spherical gap)**.** *Suppose $D \ge M \ge 6$ are fixed, M is even, and $\alpha = \alpha(D)$ is the budget in* (4.18)*. Suppose the finite coefficients satisfy*

$$\sum_{t=m}^{D} W_{mm}^{(t)} = \frac{3}{2} w_m^\infty \qquad \text{for every } m \in \mathcal{I}_M, \tag{5.7}$$

$$S(a, W, \alpha) \ge \eta h_D \qquad \text{on } \mathfrak{F}_D^{\mathrm{loc}}, \quad \eta > 0, \tag{5.8}$$

$$s := \sum_{t=1}^{D} a_t < 1 - \epsilon_M. \tag{5.9}$$

Then there exists Q_0 such that

$$H_Q^2 \ge (1 - s - \epsilon_M) H_Q \qquad \text{for every } Q \ge Q_0. \tag{5.10}$$

This is a finite-dimensional sufficient criterion. The matrices in (5.8) do not grow with Q or n. The margin η is essential for the continuity argument; positivity with an unprotected additional kernel would not suffice.

The criterion is also valid for a smaller positive four-body coefficient $0 < \alpha < \alpha(D)$: replace the finite-Q coefficient by $\alpha \alpha_Q(D)/\alpha(D)$. The saturated choice in Theorem 5.1 is the one used by the supplied certificate.

6 Kernel-preserving deformation and proof of the criterion

6.1 The spherical seed

For sufficiently large Q, all retained spherical pair and triple vectors are supported in the first $D + 1$ orbitals. Replace b_t and $C_{m,t}$ in (5.4) by $B_t^{(Q)}$ and $C_{m,t}^{(Q)}$. For each retained m, change only the diagonal coefficient at $t = m$ by

$$W_{mm}^{(m)}(Q) = W_{mm}^{(m)} + \frac{3Q - 2m + 1}{2Q - 1} w_m(Q) - \frac{3}{2} w_m^\infty. \tag{6.1}$$

All other triple entries remain fixed. Thus

$$\sum_{t=m}^{D} W_{mm}^{(t)}(Q) = \frac{3Q - 2m + 1}{2Q - 1} w_m(Q). \tag{6.2}$$

Retain all a_t and use the finite-Q four-body coefficient $\alpha_Q(D)$. Denote the resulting local operator by

$$S_Q = S_{2,3}^{(Q)}(a, W(Q)) + \alpha_Q(D) R_D^{(Q)}. \tag{6.3}$$

On the fixed space $\mathfrak{F}_D^{\mathrm{loc}}$,

$$S_Q \longrightarrow S \quad \text{in operator norm.} \tag{6.4}$$

For the pair and triple vectors, this is coefficientwise convergence in a fixed finite list, proved explicitly below and in Appendix A. For the scalar corrections it follows from (3.13) and (4.18).

Equation (6.4) alone does not establish positivity of S_Q. The limiting operator has a nontrivial kernel. We next show that the entire reference kernel is transported exactly.

6.2 An exact diagonal similarity for pair annihilation

For $0 \le a \le D$ and $Q > D$ define

$$d_a(Q) = \left[\prod_{u=0}^{a-1}\left(1 - \frac{u}{Q}\right)\right]^{1/2}, \qquad d_0(Q) = 1. \tag{6.5}$$

Let T_Q be the positive diagonal operator on $\mathfrak{F}_D^{\mathrm{loc}}$ given by

$$\mathsf{T}_Q|I\rangle = \left(\prod_{a\in I} d_a(Q)\right)|I\rangle. \tag{6.6}$$

It is invertible and $\mathsf{T}_Q \to \mathbf{1}$. Define also

$$\kappa_t(Q) = \prod_{u=2}^{t}\left(1 - \frac{u}{2Q}\right)^{-1/2}, \qquad \kappa_1(Q) = 1. \tag{6.7}$$

Lemma 6.1 (Exact reference-kernel alignment)**.** *On the local Fock space,*

$$B_t^{(Q)} = \kappa_t(Q)\,\mathsf{T}_Q^{-1} b_t \mathsf{T}_Q, \qquad 1 \le t \le D. \tag{6.8}$$

Consequently, if $h_D^{(Q)} = \sum_{t=1}^{D} (B_t^{(Q)})^\dagger B_t^{(Q)}$*, then*

$$\ker h_D^{(Q)} = \mathsf{T}_Q^{-1} \ker h_D. \tag{6.9}$$

Proof. Factor the binomial coefficients as

$$\binom{Q}{a} = \frac{Q^a}{a!} d_a(Q)^2, \qquad Q\binom{2Q-2}{t-1} = \frac{Q^t 2^{t-1}}{(t-1)!}\prod_{u=2}^{t}\left(1 - \frac{u}{2Q}\right). \tag{6.10}$$

The coefficient in (2.5) is therefore the coefficient in (5.2) multiplied by $\kappa_t(Q) d_a(Q) d_b(Q)$. Applying $\mathsf{T}_Q^{-1} c_b c_a \mathsf{T}_Q$ to an occupation vector supplies exactly $d_a(Q) d_b(Q)$, proving (6.8). The kernel of a sum of annihilator norm squares is the intersection of their kernels. The scalar factors are nonzero, giving (6.9). □

Lemma 6.2 (Every seed term annihilates the aligned kernel)**.** *For sufficiently large Q,*

$$S_Q \ker h_D^{(Q)} = \{0\}. \tag{6.11}$$

Proof. Every active triple vector is in the image of the pair–single wedge map. At total orbital degree t its annihilator is a linear combination of

$$c_a B_{t-a}^{(Q)}. \tag{6.12}$$

Since $t \le D$, every nonzero required pair index lies between 1 and D. Hence every $C_{m,t}^{(Q)}$ annihilates $\ker h_D^{(Q)}$ on every local particle-number sector. This is an identity of annihilation polynomials, not only a statement on three particles. Each product $B_s^{(Q)} B_t^{(Q)}$ with $s, t \le D$ also annihilates this kernel. In each term of S_Q, the rightmost annihilator therefore vanishes on the kernel, independently of the signs of the coefficients. □

6.3 A finite-dimensional continuity lemma

Lemma 6.3 (Positivity with an exactly fixed kernel)**.** *Let $h \geq 0$ be an operator on a fixed finite-dimensional Hilbert space, and let $S \geq \eta h$ for some $\eta > 0$. Suppose S annihilates* $\ker h$*. Let A_q be self-adjoint, $A_q \to S$ in norm, and $A_q \ker h = \{0\}$ for all q. Then $A_q \geq 0$ for all sufficiently large q.*

Proof. Let $\gamma > 0$ be the smallest positive eigenvalue of h, assuming $h \neq 0$; the case $h = 0$ and $\ker h$ equal to the entire space is immediate. On $(\ker h)^{\perp}$, $S \geq \eta\gamma \mathbf{1}$. If $\|A_q - S\| < \eta\gamma/2$, then the compression of A_q to this complement is at least $\eta\gamma/2$. Its kernel assumption and self-adjointness eliminate the two off-diagonal blocks with $\ker h$, so $A_q \geq 0$ on the entire space. □

Apply this lemma to

$$\widehat{S}_Q = \mathsf{T}_Q^{-1} S_Q \mathsf{T}_Q^{-1}. \tag{6.13}$$

Equations (6.4) and (6.5) imply $\widehat{S}_Q \to S$. Lemmas 6.1 and 6.2 give

$$\widehat{S}_Q \ker h_D = \{0\}. \tag{6.14}$$

The limiting seed also annihilates $\ker h_D$, either by the limiting polynomial relations or by taking the limit of this equality. Hypothesis (5.8) supplies the required relative margin. Therefore

$$S_Q \geq 0 \qquad \text{for all sufficiently large } Q. \tag{6.15}$$

The reference eigenvalue γ belongs to one fixed $2^{D+1} \times 2^{D+1}$ matrix. No gap of a growing many-particle system enters this argument.

6.4 Embedding, averaging, and closing the estimate

Proof of Theorem 5.1. Embed the local Fock algebra into $\mathfrak{F}_Q$. Under the factorization of Fock space for the local orbital subspace and its orthogonal complement, every even local operator is represented as its local matrix tensored with the identity. Thus (6.15), which holds in every local particle sector, remains positivity on $\mathfrak{F}_Q$. Rotating and averaging preserves it.

Use the pair and triple averaging identities (4.3)–(4.4), the diagonal constraints (6.2), and Proposition 4.2. For all sufficiently large Q,

$$\begin{aligned} 0 &\leq (2Q-1)\mathcal{T}_Q(S_Q) \\ &= sH_Q + \sum_{m \in \mathcal{I}_M} w_m(Q) K_{m,Q} + (2Q-1)\alpha_Q(D)\mathcal{T}_Q(R_D^{(Q)}) \\ &\leq sH_Q + \sum_{m \in \mathcal{I}_M} w_m(Q) K_{m,Q} + \mathcal{D}_Q. \end{aligned} \tag{6.16}$$

Every retained channel exists when Q is sufficiently large. Combining (6.16) with Proposition 3.5 gives

$$H_Q^2 \geq (1 - s - \epsilon_M) H_Q. \tag{6.17}$$

This is an operator inequality on the entire Fock space, proving the theorem. □

Remark 6.4 (Why the limit does not hide a global perturbation)**.** Only fixed-size local matrices are deformed as Q increases. We do not compare $H_{n,Q}$ with a planar Hamiltonian by an error proportional to n, nor sum local perturbation errors before proving positivity. Exact finite-Q averaging is applied only after the local operator has been proved positive. This order of operations is essential.

7 The explicit fifteen-orbital certificate

7.1 Exact active triple polynomials

All local coefficients can be specified without numerical eigenvectors. For $m \in \mathcal{I}_M$ and $a < b < c$ with $a + b + c = m$, define

$$\mathsf{q}_{m,m}(a,b,c) = \sigma_m\big[(-2)^a(c-b) - (-2)^b(c-a) + (-2)^c(b-a)\big], \tag{7.1}$$

where $\sigma_m \in \{1,-1\}$ makes the first nonzero coefficient in lexicographic order positive. Obtain $\mathsf{q}_{m,t+1}$ from $\mathsf{q}_{m,t}$ by raising each occupied orbital a to $a+1$, omitting collisions, and adding $(a+1)$ times its coefficient to the resulting increasing occupation set. No reordering sign occurs for this single adjacent increase when the target orbital is unoccupied. Equivalently, writing I and J as increasing three-element sets,

$$\mathsf{q}_{m,t+1}(J) = \sum_{\substack{|I|=3\\ \sum_{a\in I} a = t}} \mathsf{q}_{m,t}(I) \sum_{\substack{a\in I\\ a+1\notin I}} (a+1)\mathbf{1}_{\{J=(I\setminus\{a\})\cup\{a+1\}\}}. \tag{7.2}$$

Set

$$\zeta_{m,t}^2 = \frac{(m-1)!}{6^{m-1}\lambda_m^{\infty}\,3^{t-m}(t-m)!}, \qquad \zeta_{m,t} > 0. \tag{7.3}$$

Then

$$C_{m,t} = \zeta_{m,t} \sum_{\substack{a<b<c\\ a+b+c=t}} \frac{\mathsf{q}_{m,t}(a,b,c)}{\sqrt{a!b!c!}}\, c_c c_b c_a. \tag{7.4}$$

The squared norm of the vector annihilated by $C_{m,t}$ is one. It is the active eigenvector with eigenvalue λ_m^{∞} in this center-of-mass descendant sector. Appendix A derives (7.1) from the pair–single source vector and derives the normalization of its descendants.

For the retained set $D = M = 14$, there are exactly 67 vectors in (7.4). The exact auditor also verifies their normalization and eigenvector equations directly, as well as every highest-weight equation at $t = m$. These are finite rational identities, not approximate orthogonality tests.

7.2 Rational encoding of the coefficients

The normalized triple coefficients $W^{(t)}_{mm'}$ need not be rational. The stored quantities are

$$\mathsf{k}_{mm't} = W^{(t)}_{mm'}\zeta_{m,t}\zeta_{m',t}, \qquad m \le m'. \tag{7.5}$$

Each $\mathsf{k}_{mm't}$ is rational. The coefficient for the reverse order is identical. Thus the triple part of the seed can be reconstructed with rational weights on the unnormalized integer polynomials in (7.4).

Appendix B specifies every coefficient: all fourteen pair coefficients, all nonzero $\mathsf{k}_{mm't}$, and the convention that every unlisted allowed k equals zero. There are 287 independent allowed triple entries, of which 179 are nonzero. The machine-readable manifest uses the key `K` for these same rational quantities, not for the global operators $K_{m,Q}$.

Proposition 7.1 (Verified finite certificate)**.** *Take $D = M = 14$, the coefficients in Appendix B, and*

$$\alpha = \frac{8388608}{59353161}, \qquad \eta = \frac{1}{10000}. \tag{7.6}$$

Then the diagonal constraints (5.7) *hold exactly and*

$$S(a, W, \alpha) \ge \eta h_{14} \quad \text{on } \mathfrak{F}(\mathbb{C}^{15}). \tag{7.7}$$

Moreover,

$$s = \sum_{t=1}^{14} a_t = \frac{117142992187}{125000000000}. \tag{7.8}$$

Proof. The trace identities, pair cost, and four-body budget are exact rational checks from the displayed coefficient definitions. The positive-semidefinite inequality is established by the exact assembly and interval verification in Section 8. The verification covers every occupation state of the fifteen-orbital Fock space, including high local particle numbers. Its logically sufficient tests, exact input-data identifiers, and fresh execution results are given there. □

7.3 Conclusion of the spectral-gap argument

Proof of Theorem 1.1 and Corollary 1.2. Proposition 7.1 provides the hypotheses of Theorem 5.1. The budget coefficient agrees with (4.19). Exact arithmetic gives

$$1 - \frac{117142992187}{125000000000} - \frac{193}{4096} = \frac{15736921879}{10^{12}} > \frac{3}{200} > 0. \tag{7.9}$$

Thus (1.4) holds for all sufficiently large Q. If $E > 0$ is an eigenvalue, $E^2 \geq g_* E$ gives $E \geq g_*$.

At $Q_N = 3(N-1)$, Proposition 2.2 identifies the unique zero mode. For all sufficiently large N, $\Delta_N \geq g_*$. For each remaining $N \geq 2$, the Hamiltonian is a nonzero finite-dimensional positive operator with a one-dimensional kernel, hence $\Delta_N > 0$. The minimum of this finite list with g_* is positive, proving (1.6). □

The argument uses no assumption about screening, decay of correlations, or a spectral gap of an effective theory. The positive constant is a conservative certified bound, not an estimate of the actual excitation energy.

8 Exact assembly and rigorous verification of the certificate

This section specifies a finite computation sufficient to prove Proposition 7.1. The role of the computation is limited to the coefficient identities and the matrix inequality (7.7). In particular, the verifier does not extrapolate a spectrum, assume positivity from rounded eigenvalues, or test only low particle numbers.

8.1 Inputs and the operator being verified

The primary mathematical input is the coefficient manifest `rational_candidate.json`. It contains the rational values of a_t, $\mathsf{k}_{mm't}$, α, and η in Section 7.1. All of them are also specified in Appendix B. The asserted matrix inequality is

$$A^{\mathrm{phys}} := S - \eta h_{14} \geq 0. \tag{8.1}$$

Thus the pair coefficient entering this matrix is $a_t - \eta$, not a_t. The triple and four-body coefficients are unchanged.

Three auxiliary files accelerate verification. They are not extra mathematical assumptions: `fewbody_integers.txt` supplies a proposed normally ordered expansion, `blocks.txt` supplies a proposed block partition and coordinate complements, and `inverse_cholesky.bin` supplies approximate positive-definiteness witnesses. Their consistency or sufficiency is checked as described below. Exact hashes identifying the supplied inputs appear in Appendix C.

8.2 An independent rational reconstruction

The Python auditor uses integer arithmetic and `fractions.Fraction`. It independently performs the following finite identities before the positive-semidefinite verification.

(i) It generates the integer highest-weight and descendant polynomials from (7.1)–(7.2), verifies all 67 normalizations and active eigenvector equations, and verifies the highest-weight equations.

(ii) It checks the eleven diagonal trace constraints (5.7), the exact pair cost (7.8), and the budget values in Table 1 using the finite sums (4.16).

(iii) It constructs the two-, three-, and four-body coefficient matrices of A^{phys} directly from the manifest, including both triple off-diagonal orders and the ordered four-body pair sum. Every resulting rational entry must agree exactly with the proposed integer expansion after multiplication by its common denominator.

There are 17 108 nonzero entries in the combined homogeneous few-body expansion. These are matrix entries before lifting to the complete Fock space, not a count of the final Fock-space nonzeros. The auditor rejects a run with Python optimization enabled, because the implementation uses exact assertions as checks. No numerical optimization package is needed to execute it.

8.3 Factorial congruence and integer Fock matrices

Here is the cancellation that makes exact assembly inexpensive. For an occupation set I, define

$$f_I = \prod_{a \in I} a!, \qquad f_\varnothing = 1, \qquad T_{\mathrm{fac}}|I\rangle = \sqrt{f_I}|I\rangle. \tag{8.2}$$

Every normally ordered coefficient of A^{phys} has the form

$$\frac{F_{pq}}{\sqrt{f_p f_q}}\, c_q^\dagger c_p, \qquad F_{pq} \in \mathbb{Q}, \qquad |p| = |q| \in \{2, 3, 4\}. \tag{8.3}$$

The factorials from an intermediate pair partition cancel in the four-body product because its two annihilated pairs are disjoint; coefficients of coincident orbital indices vanish by the anticommutation relations.

Let $I = p \cup R$ be an occupied input, with $p \cap R = \varnothing$. A term in (8.3) contributes only if $q \cap R = \varnothing$, and then the output is $J = q \cup R$. Let $\sigma(R, p)$ denote the fermionic sign of removing p from $p \cup R$. The corresponding entry in the congruent matrix is

$$\langle J|T_{\mathrm{fac}} A^{\mathrm{phys}} T_{\mathrm{fac}}|I\rangle = \sigma(R, p)\sigma(R, q) F_{pq} f_R, \tag{8.4}$$

with summation if several terms contribute. To see the cancellation, use $\sqrt{f_I f_J}/\sqrt{f_p f_q} = f_R$. A single common positive denominator G therefore makes

$$A = G T_{\mathrm{fac}} A^{\mathrm{phys}} T_{\mathrm{fac}} \tag{8.5}$$

a real symmetric integer matrix. Positivity of A and of A^{phys} are equivalent. The C++ verifier reconstructs A with GMP integers using (8.4); it also checks symmetry exactly.

8.4 Complete block coverage

Particle number n and total orbital degree $t = \sum_{a \in I} a$ commute with the seed. They decompose the local Fock space into blocks. The verifier enumerates the proposed occupation masks and checks that every mask with at least two particles occurs exactly once, that no mask occurs twice, and that the particle number and degree agree with the advertised block. It checks that every assembled nonzero entry remains inside that block.

The vacuum and fifteen one-particle states are annihilated by every term of A^{phys} by its body order. The other 32 752 states occur in 560 blocks, the largest of dimension 289. Thus no high-density local sector is omitted. Table 2 gives the accounting by particle number.

Table 2: Exhaustive local Fock-space coverage by particle number. "Positive rank" sums the certified complement dimensions, not a rank inferred from rounded eigenvalues. Vacuum and one-particle sectors vanish by body order.

Particles	Blocks	States	Positive rank	Largest block
2	27	105	14	7
3	37	455	169	25
4	45	1365	870	69
5	51	3003	2541	141
6	55	5005	4795	227
7	57	6435	6399	289
8	57	6435	6434	289
9	55	5005	5005	227
10	51	3003	3003	141
11	45	1365	1365	69
12	37	455	455	25
13	27	105	105	7
14	15	15	15	1
15	1	1	1	1
Total	560	32752	31171	289

8.5 Exact kernels and coordinate complements

For each nonzero block, let d be its dimension. The input file proposes a set P of r coordinate indices and, when $r < d$, row indices in a pair-removal matrix. The remaining coordinates are denoted P^c.

The reference pair-removal matrix can be written with integer entries $\sigma(R, \{a, b\})(b - a)$ after the same factorial change of variables and an independent positive row rescaling. The total degree fixes the removed pair index. Fraction-free Gaussian elimination with exact divisions constructs $d - r$ candidate integer vectors z_ν from the selected pair-removal rows. Their P^c coordinates form a nonsingular diagonal matrix: each vector has one nonzero free coordinate and all other free coordinates zero. The program checks all selected coordinate indices and all divisions.

Crucially, the pair-removal calculation is only a way of proposing vectors. The verifier then checks

$$Az_\nu = 0 \quad \text{in every row of the complete integer block,} \qquad \nu = 1, \dots, d - r. \tag{8.6}$$

It does not infer these identities from a small floating-point residual. Since the vectors z_ν, together with the coordinate vectors indexed by P, form a basis, the block is congruent to

$$\begin{pmatrix} A_{PP} & 0 \\ 0 & 0 \end{pmatrix}. \tag{8.7}$$

It remains to prove positive definiteness of A_{PP}. If $r = d$ there is no kernel step. If $r = 0$, every entry of the block is checked to be exactly zero.

This procedure neither assumes a numerical rank nor assumes in advance that the nullspace of $S - \eta h_{14}$ equals that of h_{14}. The identities and the complete spanning set in (8.7) are sufficient for the actual matrix under test.

8.6 Outward-rounded positive-definiteness test

For a positive-rank block, divide A_{PP} by G and apply a diagonal scaling by integer powers of two. This gives an exact rational real symmetric matrix M. The scaling is positive and invertible; it changes

neither positivity nor the need for proof. The supplied binary file contains a real square matrix F of IEEE binary64 numbers, intended to precondition M. Each binary64 number is treated as an exact dyadic rational.

The following elementary criterion is the only positive-definiteness test needed.

Lemma 8.1 (Interval congruence certificate)**.** *Let M be real symmetric and let F be any real square matrix. Suppose entrywise intervals $[L_{ij}, U_{ij}]$ contain $C = F^{\mathsf{T}}MF$, and a certified upper bound satisfies*

$$\rho \geq \max_i \sum_j \max\{|L_{ij} - \delta_{ij}|, |U_{ij} - \delta_{ij}|\}, \qquad \rho < 1. \tag{8.8}$$

Then M is positive definite.

Proof. The exact matrix $C - \mathbf{1}$ is symmetric, so its spectral norm is at most its maximum absolute row sum, which is at most ρ. Therefore $C \geq (1 - \rho)\mathbf{1} > 0$. This implies that F is nonsingular, and congruence by F^{-1} gives $M > 0$. □

The validity of the test does not depend on F actually being an inverse Cholesky factor. A poor F merely causes the test to fail; it cannot make an indefinite exact M pass.

Each rational entry of M is enclosed by two binary64 endpoints. Their ordering relative to the exact rational is checked using GMP. The products MF and $F^{\mathsf{T}}(MF)$ are then bounded by interval matrix multiplication, with the rounding mode set downward for lower endpoints and upward for upper endpoints. Multiplication by a negative entry of F reverses the choice of endpoint. Volatile intermediate multiplication and addition prevent reassociation or contraction; the compiler flags prohibit fast-math transformations and fused multiply–add contraction. The final maximum absolute row sum is accumulated with upward rounding.

The verifier checks IEEE binary64 arithmetic, the endianness of the witness file, both directed rounding modes, and subnormal behavior. It rejects nonfinite endpoints and failed enclosure comparisons. The delivered implementation was built with the flags in Appendix C. Consequently, the printed decimal residual alone is not the certificate: the certificate is the exact input together with the checked interval enclosure and the strict rational inequality $\rho < 1$.

8.7 Recorded results and their logical meaning

A fresh execution of the manifest auditor and the directed-interval verifier produced the accounting in Table 3. The largest interval radius is exactly the dyadic rational

Table 3: Complete finite verification of $S - 10^{-4}h_{14} \geq 0$. The error radius is a preconditioned matrix-test quantity, not a spectral-gap estimate.

Quantity	Verified value
Local orbital count	15
Full Fock dimension	32768
Checked dimension, excluding trivial sectors	32752
Particle-number/degree blocks	560
Positive-rank blocks	511
Exactly zero blocks	49
Largest block dimension	289
Explicit null vectors in positive-rank blocks	1366
Certified relative seed margin η	1/10000

$$\rho_{\max} = \frac{6173581662907375}{1208925819614629174706176} < 5.107 \times 10^{-9} < 1. \tag{8.9}$$

All full block kernels are covered by (8.6); the 49 zero blocks are checked by entrywise integer equality. Lemma 8.1 proves positive definiteness on every selected coordinate complement. The block coverage and the trivial vacuum/one-particle observation prove (8.1), completing the computational proof of Proposition 7.1.

An additional implementation based on explicit floating-point forward-error bounds, rather than changing the rounding mode, also verifies the same rational principal blocks. Its maximum certified radius is less than 5.444×10^{-7}. This second calculation is a useful arithmetic cross-check. The directed-interval calculation above is the one used for the certificate stated in this paper.

8.8 Trusted components and limits of the verification claim

The positive-semidefinite claim is computer assisted. The trusted components are the analytic identification of the seed, the correctness of the small exact-reconstruction and verification programs, arbitrary-precision integer and rational arithmetic, and the specified floating-point behavior of the compiled interval operations. Exact file hashes identify the computation but do not by themselves establish its correctness. The rational reconstruction and explicit source code allow a different implementation to be written without using the numerical optimizer that originally produced the coefficients.

The verification is not a proof-assistant formalization, nor is the presence of a second arithmetic implementation a claim of independent authorship or independent referee replication. The displayed mathematical reductions remain subject to the same scrutiny as any new proof. In particular, the all-particle-sector normalization of (2.11), the uniform channel estimate (3.19), and the exact-kernel argument in Section 6 are analytic statements, not consequences of finite numerical testing.

9 Consequences, limitations, and reproducibility

The quantitative conclusion is an operator inequality for the full spherical pseudopotential. It is stronger than a statement about a finite list of ground-state energies, and weaker than a numerical lower bound for every finite N: the local-to-sphere argument provides the existence of a finite threshold, not its explicit value. A computation of that threshold would require effective bounds on the fixed-window coefficient convergence and a certified lower bound for the positive spectrum of h_{14}. Neither is needed for uniform positivity in Corollary 1.2.

The method uses no screening estimate for the Laughlin plasma. It does not infer dynamics from incompressibility or from the exact form of the zero mode. Its quantitative input is a finite positive-semidefinite inequality, with the infinite angular-channel tail controlled analytically. The four-particle term in H^2 supplies positivity in densely occupied local sectors; antisymmetry removes the two most expensive auxiliary pair-pair channels before averaging. These two facts account for the improvement over a comparison that discards the entire disjoint-pair remainder.

Several further statements would require separate arguments. A gap for another geometry is not obtained simply by changing the single-particle basis. A perturbative stability theorem would need suitable locality and topological-order hypotheses. A Coulomb interaction is not bounded below by the parent interaction in a way that identifies ground spaces or excitation gaps automatically. Likewise, the thermodynamic spectral bound alone is not a construction of every desired infinite-volume observable. None of these statements is included in Theorem 1.1.

The finite-certificate theorem is reusable independently of the particular coefficient set. Other orbital windows, angular cutoffs, and local positive operators may improve the lower bound or apply to related

interactions. The theorem is sufficient, not necessary: failure of a chosen local search would not imply gaplessness.

Data and code availability. The accompanying verification directory contains the exact coefficient manifest, all auxiliary inputs, the rational auditor, both positive-definiteness implementations, block-by-block outputs, and additional exact consistency tests. The essential coefficients are reproduced in this manuscript. Appendix C gives input hashes, file formats, and commands for rerunning the proof-relevant computation. No optimizer, network service, proprietary solver, or hidden numerical data are required for verification. The manuscript and source package have no assigned archival repository identifier at the time of this version; permanent deposition and bibliographic metadata should accompany public submission.

A Explicit fixed-window convergence of the angular vectors

This appendix supplies coefficient-level details for the convergence used in Section 6. Only a fixed finite set of magnetic deficits is involved. In particular, no limit of an operator on an expanding Fock space is asserted.

A.1 The pair–single highest-weight vector

Fix an active m, put $k = m - 1$, and take Q sufficiently large for all displayed indices. A highest-weight vector of total spin $L_m = J + j - k$ in $V_J \otimes V_j$ has the form

$$\eta_{m,Q} = \sum_{p=0}^{k} A_p(Q)|J, J-p\rangle \otimes |j, j-(k-p)\rangle. \tag{A.1}$$

The equation $(J_+ \otimes \mathbf{1} + \mathbf{1} \otimes j_+)\eta_{m,Q} = 0$ gives

$$\frac{A_{p+1}(Q)}{A_p(Q)} = -\sqrt{\frac{(k-p)(Q-k+p+1)}{(p+1)(2Q-2-p)}}, \qquad 0 \le p < k. \tag{A.2}$$

Choose the overall phase and normalize the vector. At fixed k, every coefficient converges, and one possible limiting phase convention is

$$A_p(\infty) = (-1)^p \sqrt{\binom{k}{p}\frac{2^{k-p}}{3^k}}. \tag{A.3}$$

The squared coefficients sum to one by the binomial theorem.

Apply the wedge map T_Q to (A.1). Its squared norm is $\lambda_m(Q)$ by the three-particle Gram calculation, so the normalized active highest-weight vector is $T_Q\eta_{m,Q}/\sqrt{\lambda_m(Q)}$. The pair vector at deficit p is annihilated by $B^{(Q)}_{p+1}$. Its coefficients converge to those of b_{p+1}. Combining the three possible choices of the remaining single orbital in the wedge gives, for $a < b < c$ and $a + b + c = m$, a coefficient proportional to

$$\frac{(-2)^a(c-b) - (-2)^b(c-a) + (-2)^c(b-a)}{\sqrt{a!b!c!}}. \tag{A.4}$$

Its common normalization factor, before the harmless sign choice in (7.1), is

$$\sqrt{\frac{(m-1)!}{6^{m-1}\lambda_m^\infty}}. \tag{A.5}$$

This proves the $t = m$ case of (7.4). The polynomial is nonzero for each active m because $\lambda_m^\infty > 0$.

A.2 Center-of-mass descendants

Let $d = t - m$. Normalized total-spin lowering in the spherical active multiplet gives

$$v_{m,t}^{(Q)} = \frac{(L_-)^d}{\sqrt{d!\,(2L_m)(2L_m-1)\cdots(2L_m-d+1)}} v_{m,m}^{(Q)}. \tag{A.6}$$

The single-particle lowering operator obeys

$$Q^{-1/2} L_- e_a = \sqrt{(a+1)(1-a/Q)}\, e_{a+1}. \tag{A.7}$$

For fixed occupied orbitals this converges to the usual oscillator creation coefficient $\sqrt{a+1}$. Since $2L_m = 3Q - 2m$, the denominator in (A.6), after dividing by $Q^{d/2}$, tends to $\sqrt{3^d d!}$. Acting on coefficients of the form $q(a,b,c)/\sqrt{a!b!c!}$, oscillator raising replaces an occupied a by $a+1$ and multiplies the integer numerator by $a+1$. Terms producing repeated occupied orbitals vanish. This is precisely the recurrence (7.2).

It follows that

$$\zeta_{m,t}^2 = \frac{(m-1)!}{6^{m-1}\lambda_m^{\infty} 3^{t-m}(t-m)!}, \tag{A.8}$$

and that every coefficient of $v_{m,t}^{(Q)}$ converges to the vector annihilated by $C_{m,t}$ in (7.4). For fixed D, only finitely many (m,t) and occupation triples occur. The corresponding annihilation operators therefore converge in norm on $\mathfrak{F}_D^{\mathrm{loc}}$.

This also identifies the kernel property used in the main proof. Each normalized active spherical vector is in the range of the pair–single wedge map. At total degree t its annihilator is consequently a linear combination of $c_a B_{t-a}^{(Q)}$. When $t \le D$, all nonzero pair indices in this expression are at most D. The common reference kernel is therefore annihilated exactly, not only in the planar limit.

A.3 The two-pair highest-weight vector and budget limit

For two equal spin-J representations, a highest-weight vector of deficit r has coefficients $A_p^{\mathrm{pp}}(Q)$, $0 \le p \le r$, satisfying

$$\frac{A_{p+1}^{\mathrm{pp}}(Q)}{A_p^{\mathrm{pp}}(Q)} = -\sqrt{\frac{(r-p)(2J-r+p+1)}{(p+1)(2J-p)}}. \tag{A.9}$$

As $Q \to \infty$ with r fixed, its normalized coefficients converge, up to a common phase, to those of $(x-y)^r/\sqrt{2^r r!}$ in the Bargmann basis. Applying d normalized total-spin lowerings produces the factor $(x+y)^d/\sqrt{2^d d!}$. This proves (4.14) directly.

Projection to pair deficits at most $D-1$ restricts the total deficit to $r + d \le 2D - 2$. Thus the trace of the projected finite-spin multiplet is a finite sum of squared coefficients converging to (4.16). Since all allowed limiting $k_r(14)$ in Table 1 are strictly positive, every relevant finite-Q $k_r(Q,14)$ is positive for sufficiently large Q. Taking the minimum over this fixed finite set proves (4.18). No estimate uniform in unbounded r is used here; such uniformity is supplied separately by the tail bound of Section 3.

B Complete coefficient specification

The following tables, together with the polynomial recurrence in Section 7.1, specify the local seed without referring to numerical search output. The fixed parameters are

$$D = M = 14, \qquad \eta = \frac{1}{10000}, \qquad \alpha = \frac{8388608}{59353161}. \tag{B.1}$$

Table 4: All pair coefficients a_t. The common certified seed margin is subtracted only when assembling $S - \eta h_{14}$.

t	a_t	t	a_t
1	$\frac{72764512071}{200000000000}$	8	$\frac{10082575077}{1000000000000}$
2	$\frac{233875687859}{1000000000000}$	9	$\frac{4049997117}{500000000000}$
3	$\frac{15656664701}{125000000000}$	10	$\frac{81}{10000}$
4	$\frac{32413859957}{500000000000}$	11	$\frac{81}{10000}$
5	$\frac{5610905959}{250000000000}$	12	$\frac{81}{10000}$
6	$\frac{60138458613}{1000000000000}$	13	$\frac{81}{10000}$
7	$\frac{81}{10000}$	14	$\frac{81}{10000}$

The pair entries in Table 4 are a_t in S, not $a_t - \eta$ in the matrix A^{phys} tested for positivity. Their sum is (7.8).

Table 5 lists every nonzero $\mathsf{k}_{mm't}$ for $m \le m'$. Each row gives the numerator and the positive denominator of the rational number. Every unlisted allowed entry is exactly zero. For $m > m'$, set $\mathsf{k}_{mm't} = \mathsf{k}_{m'mt}$. Recover the normalized coefficients through

$$W^{(t)}_{mm'} = \frac{\mathsf{k}_{mm't}}{\zeta_{m,t}\zeta_{m',t}}. \tag{B.2}$$

The full operator sum includes both orders of an off-diagonal entry. The convention for the signs of the integer triple vectors is the lexicographically first nonzero coefficient being positive, as in (7.1); altering that convention requires conjugating the corresponding rows and columns of $W^{(t)}$.

Table 5: Complete nonzero rational triple coefficients $\mathsf{k}_{mm't}$. All unlisted allowed entries are zero; reverse indices are set by symmetry.

t	m	m'	Numerator	Denominator
3	3	3	1582418587181	928455029464035206174343168
5	5	5	870367236479	69634127209802640463075737 6
6	3	3	4472293539495689	59029581035870565 1712
6	5	5	2384245904490639	2305843009213693952
6	5	6	−2775481686851229	9223372036854775808
6	6	6	−25354700736472426801320 53	10445119081470396069461360 64
7	3	3	6574501973153831	4722366482869645213696
7	5	5	6761105355803023	73786976294838206464
7	5	6	−2383670722093247	18446744073709551616
7	6	6	−2461256953732705	4611686018427387904
7	6	7	−2441423081908847	36893488147419103232
7	7	7	2628719515	16320498564797493858533376
8	3	3	6045711795882965	37778931862957161709568
8	3	6	−6184717990864363	4722366482869645213696
8	3	8	2271414123357001	2361183241434822606848
8	5	5	6294197500014455	1180591620717411303424
8	5	6	−5930536067088493	295147905179352825856
8	6	6	−408971136365481	4611686018427387904
8	6	7	−2154152132080945	147573952589676412928
8	6	8	−2628798951439627	36893488147419103232

Continued on the next page.

Table 5 (continued): exact triple coefficients.

t	m	m'	Numerator	Denominator
8	7	7	193130636664041	576460752303423488
8	7	8	−6192824617041133	9223372036854775808
8	8	8	−4764882536136736669979	6120186961799060196950016
9	3	6	−5280906402335913	151115727451828646838272
9	5	6	−7092702778727079	4722366482869645213696
9	5	8	4671063166525573	1180591620717411303424
9	5	9	−7206744038754075	2361183241434822606848
9	6	6	−4434950410382545	590295810358705651712
9	6	7	−2874505618370783	590295810358705651712
9	6	8	−2639229110186847	295147905179352825856
9	6	9	3255377441921395	147573952589676412928
9	7	7	6185399330804767	73786976294838206464
9	7	8	−6893446898455541	147573952589676412928
9	7	9	1556561253994693	9223372036854775808
9	8	8	−5702642829373067	18446744073709551616
9	8	9	−5982041302890025	73786976294838206464
9	9	9	2829684262990651009427	9180280442698590295425024
10	3	3	6021599260120797	4835703278458516698824704
10	3	8	−4994121612248647	151115727451828646838272
10	3	10	4639996480694933	18889465931478580854784
10	5	5	2547758693934293	37778931862957161709568
10	5	6	−3790833577277541	37778931862957161709568
10	6	6	−4474555431926563	944473296573929042739 2
10	6	7	−4701541174192767	18889465931478580854784
10	6	8	−6393471263760701	4722366482869645213696
10	6	9	−2895508647961791	1180591620717411303424
10	6	10	1522045077346371	147573952589676412928
10	7	8	−6110019355099049	590295810358705651712
10	7	10	7733205692512963	295147905179352825856
10	8	8	−6284841628655667	147573952589676412928
10	8	9	−1947615688759465	147573952589676412928
10	8	10	−8964001798010779	590295810358705651712
10	9	9	6212229486439287	73786976294838206464
10	9	10	−8045857678365293	36893488147419103232
10	10	10	−32681361287795522935	23906980319527578894336
11	3	3	6329287384126699	154742504910672534362390528
11	5	5	6480742475695759	4835703278458516698824704
11	5	6	−7515834043009503	1208925819614629174706176
11	5	8	−2205645101206755	151115727451828646838272
11	6	6	−1260036628166277	37778931862957161709568
11	6	7	−5321280538887723	151115727451828646838272
11	6	8	−7966020412933223	75557863725914323419136
11	6	10	6843890667263889	18889465931478580854784
11	6	11	6037472957241103	151115727451828646838272
11	7	7	5249818999805325	944473296573929042739 2
11	7	8	−5910194557433919	4722366482869645213696
11	7	9	2405127673196307	2361183241434822606848
11	8	8	−314015515605467	73786976294838206464
11	8	9	−3404863476411487	590295810358705651712
11	8	10	−2494332393150189	295147905179352825856
11	9	9	1890663100682521	73786976294838206464
11	9	10	−6688494797466931	73786976294838206464
11	10	10	−1941266158401333	18446744073709551616
11	10	11	3497399370730401	18446744073709551616

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Table 5 (continued): exact triple coefficients.

t	m	m'	Numerator	Denominator
11	11	11	44609965	71720940958582736683008
12	3	3	5002941296633473	49517601571415210995964968 96
12	3	10	3519280383138081	604462909807314587353088
12	5	5	2594601299286201	967140655691703339764940 8
12	5	6	−5826972070490973	773712524553362671811952 64
12	5	8	−7328249066833811	967140655691703339764940 8
12	6	6	−5286537514643669	483570327845851669882470 4
12	6	7	−1120971486149199	604462909807314587353088
12	6	8	−966289721113637	75557863725914323419136
12	6	9	−4921089509187963	302231454903657293676544
12	6	10	234045735910123	4722366482869645213696
12	6	12	−340534930993325	1180591620717411303424
12	7	7	3528827690855081	151115727451828646838272
12	7	8	−6369498851268745	75557863725914323419136
12	7	10	2576779821155705	9444732965739290427392
12	8	8	−129933203657001	59029581035870565 1712
12	8	10	−1702947946057163	1180591620717411303424
12	8	11	8469863767364369	2361183241434822606848
12	8	12	5402489353448899	4722366482869645213696
12	9	9	4271864886367793	59029581035870565 1712
12	9	10	−22158347903645	2305843009213693952
12	9	12	−4817329311155481	1180591620717411303424
12	10	10	−799150604508267	36893488147419103232
12	10	12	−5355843042576495	36893488147419103232
12	11	11	5556525522774971	73786976294838206464
12	11	12	−6120224599877435	9223372036854775808
12	12	12	−66290406884338495885	26895352859468526256128
13	3	3	3239031837701343	39614081257132168796771975168
13	3	7	5474662578974213	12379400392853802748991242 24
13	3	12	−3495818894435457	2417851639229258349412352
13	5	5	8752488899240967	61897001964269013744956211 2
13	6	6	−3157285071886465	773712524553362671811952 64
13	6	7	−4768364869503229	15474250491067253436239052 8
13	6	8	−7135181378650451	19342813113834066795298816
13	6	10	4629454650887819	1208925819614629174706176
13	6	11	3646314844618131	604462909807314587353088
13	6	12	4803773991524239	604462909807314587353088
13	6	13	−5293238011246159	302231454903657293676544
13	7	7	5591658642269303	19342813113834066795298816
13	7	8	−1832046356930341	604462909807314587353088
13	7	11	−6998163647655987	302231454903657293676544
13	7	12	6179160883353915	75557863725914323419136
13	8	8	−5950917116428629	604462909807314587353088
13	8	11	−6221750477964701	18889465931478580854784
13	8	12	−6423103986018975	75557863725914323419136
13	8	13	3913730572598549	2361183241434822606848
13	9	9	7508673051526781	37778931862957161709568
13	9	10	−2837112808618279	2361183241434822606848
13	9	12	2103379801542549	1180591620717411303424
13	10	10	−7945196515998759	2361183241434822606848
13	11	11	8455821609058731	147573952589676412928
13	11	12	−1743860028234753	9223372036854775808
13	11	13	−2485196212709793	18446744073709551616
13	12	13	−1241049561867063	73786976294838206464
13	13	13	−43	210119944214597861376

Continued on the next page.

Table 5 (continued): exact triple coefficients.

t	m	m'	Numerator	Denominator
14	3	3	518585270701889	7922816251426433759354395033 6
14	3	6	1174180223310901	1584563250285286751870879006 72
14	3	8	−2509743393964733	7922816251426433759354395033 6
14	3	11	−2428396833203631	3868562622766813359059763 2
14	5	5	5704935859035665	9903520314283042199192993792
14	5	6	−6984962001259747	1980704062856608439838598758 4
14	5	7	−2306078372231151	6189700196426901374495621 12
14	5	8	1745081894888329	6189700196426901374495621 12
14	5	10	−4842862259718197	1547425049106725343623905 28
14	5	11	6925152772270363	4835703278458516698824704
14	5	12	−2884829806456533	2417851639229258349412352
14	5	14	2607753813743059	75557863725914323419136
14	6	6	−6892739037136587	4951760157141521099596496896
14	6	7	7391972849185701	1237940039285380274899124224
14	6	8	−2783849765723139	1547425049106725343623905 28
14	6	9	4096128635950393	3094850098213450687247810 56
14	6	11	−5586257347500047	9671406556917033397649408
14	6	12	4551310784142091	9671406556917033397649408
14	6	14	−471340907262087	37778931862957161709568
14	7	7	3020984042573811	1934281311383406679529881 6
14	7	8	−7806730084711163	3868562622766813359059763 2
14	7	9	−3328774944606167	3868562622766813359059763 2
14	7	10	7692560488205561	1934281311383406679529881 6
14	7	13	−8512541657094587	75557863725914323419136
14	7	14	−5400244952660103	6044629098073145873530 88
14	8	8	−197360845034657	3022314549036572936765 44
14	8	9	4395680826527639	2417851639229258349412352
14	8	10	−1263375107482297	1208925819614629174706176
14	8	12	8564828555063925	1511157274518286468382 72
14	8	14	−2293773541911943	2361183241434822606848
14	9	9	6782026383372915	3022314549036572936765 44
14	9	10	−5661138684119555	1511157274518286468382 72
14	9	11	−1772502329882599	4722366482869645213696
14	9	12	5940177498802289	18889465931478580854784
14	9	13	1540589172284527	59029581035870565 1712
14	9	14	−762768976534425	73786976294838206464
14	10	10	−607632012701951	9444732965739290427392
14	10	11	1836757123617467	18889465931478580854784
14	10	12	188266985624203	59029581035870565 1712
14	10	13	6067779042102923	59029581035870565 1712
14	10	14	8098344676065149	59029581035870565 1712
14	11	11	4708214652783617	59029581035870565 1712
14	11	12	−7245743416665135	295147905179352825856
14	11	13	−5090888283653407	147573952589676412928
14	11	14	1952699167768071	9223372036854775808
14	12	12	−467158569908115	73786976294838206464
14	12	14	−4138750001687531	9223372036854775808
14	13	13	270981603736289	288230376151711744
14	13	14	−3530139864443285	1152921504606846976
14	14	14	−475475	143327232

C Reproduction, input identities, and auxiliary checks

C.1 Minimal reproduction command

The supplement is arranged as

```
verification/
  run_verification.sh
  certificate/
    rational_candidate.json
    fewbody_integers.txt
    blocks.txt
    inverse_cholesky.bin
  scripts/
    audit_exact_manifest.py
    validate_certificate_interval.cpp
    validate_certificate_gamma.cpp
  results/
  audit/
```

The proof-relevant verification requires Python 3 with its standard library, a C++17 compiler, and GMP with its C++ interface. On a supported little-endian IEEE binary64 system, run

```
cd verification
bash run_verification.sh
```

The script first reconstructs and audits the exact manifest. It then compiles and executes the interval verifier using

```
g++ -O3 -std=c++17 -fno-fast-math -ffp-contract=off \\
    -frounding-math scripts/validate_certificate_interval.cpp \\
    -lgmpxx -lgmp -o verifier
./verifier certificate
```

The temporary executable made by the supplied shell script is removed on exit. The generated JSON and TSV files are verification outputs; a preexisting success file is never used as an input to decide positivity. A failed requirement produces a nonzero exit status.

The fresh execution summarized in Section 8 used Python 3.13.5, GNU C++ 14.2.0, GMP 6.3.0, and a little-endian x86_64 host. The source explicitly tests the floating-point properties needed by its interval operations. Reproduction on a different system must pass the same checks; unsupported behavior is not silently accepted.

C.2 Exact input hashes and file formats

Table 6 gives the SHA-256 identifiers of the four immutable inputs. They fix a particular certificate and its supplied numerical witnesses. Auxiliary test outputs may change in formatting or timing between runs; their essential exact inequalities are checked afresh.

The manifest stores rational numbers as numerator/denominator strings and the normalization factors described in Section 7.1. The few-body file begins with the orbital cutoff, a common positive denominator, and the number of input-mask groups. Each group lists an annihilated occupation mask and the created masks with integer coefficients. The independently assembled entries are checked before this compressed representation is used.

The block file specifies (n, t, d, r), all d occupation masks, the r selected coordinate indices, and, for a singular positive-rank block, the selected pair-removal row indices. The verifier checks the indices and coverage, reconstructs full exact matrices, constructs the proposed null vectors, and checks their

Table 6: SHA-256 identifiers of the immutable certificate inputs. Each hash occupies two lines; concatenate them without a space.

Input	SHA-256
`rational_candidate.json`	`9d600083945ff17c6c9603a60e1b9d88` `af97dec638c055c6241d89b00683bc4c`
`fewbody_integers.txt`	`a561771fb4369e5a09a43b99300eaf72` `0a61fd08b7af7c84bcebd6bb473aa355`
`blocks.txt`	`bd51b73037f38d6774edf135eed09e82` `357ea56ff71926613ed76423e5297a70`
`inverse_cholesky.bin`	`27f92aa838c3e9b3ee921bad3af22206` `6719447c13f6d1d55314945c6f4e3b33`

products with the full matrices. Thus no asserted rank in the block file is trusted without sufficient exact and positive-definiteness checks.

The binary file contains the row-major, little-endian binary64 preconditioner matrices for the positive-rank blocks, concatenated in block order. The verifier rejects a short file, a nonfinite entry, or trailing unused witnesses. These matrices need not be regenerated with the same numerical software: any replacements that pass Lemma 8.1 prove the same rational principal-block inequalities.

C.3 Additional exact consistency checks

The supplement contains two additional tests performed during preparation of the manuscript. They support the normalization audit but are not used in place of the analytic proofs for arbitrary flux.

First, the displayed three-particle formula and the reduced Racah expression were each checked against exact Wigner $6j$ evaluation for every $Q = 6, \dots, 24$ and $m = 3, \dots, Q + 1$. There are 266 exact comparisons for each expression. They all agree, including integral and half-integral orbital spins and the endpoint with one Racah summand. The optional script uses SymPy; SymPy is not a dependency of the proof-relevant finite certificate verifier.

Second, the complete square identity was independently assembled with rational matrices on the four-particle spaces at $Q = 6, 7, 8$, of dimensions $35, 70, 126$. The pair Hamiltonian, the lift of $H_{3,Q}^2 - H_{3,Q}$, and the ordered disjoint-pair term were constructed separately. All 22 001 entries of the three resulting difference matrices are exactly zero. This checks both shared-particle normalization and the factor of two for unequal pair-removal indices. The test does not replace Proposition 2.3; it is a finite diagnostic against an implementation or convention mismatch.

C.4 Audit priorities

An independent mathematical review can proceed without repeating the coefficient search. The analytic proof reduces the conclusion to a finite matrix statement; a separate exact or interval implementation can verify that statement from Appendix B. Particular care is warranted at five points: the lift normalization in (2.4), the ordered four-body identity (2.11), the particle-number-independent estimate (3.19), removal of the pair-pair source channels $r = 0, 2$, and the exact common kernel in the local-to-sphere deformation. These points are stated and proved separately precisely because a finite spectrum alone could not validate them.

References

[1] F. D. M. Haldane. Fractional quantization of the hall effect: A hierarchy of incompressible quantum fluid states. *Physical Review Letters*, 51:605–608, 1983. doi:10.1103/PhysRevLett.51.605.

[2] S. Knabe. Energy gaps and elementary excitations for certain VBS-quantum antiferromagnets. *Journal of Statistical Physics*, 52:627–638, 1988. doi:10.1007/BF01019721.

[3] R. B. Laughlin. Anomalous quantum hall effect: An incompressible quantum fluid with fractionally charged excitations. *Physical Review Letters*, 50:1395–1398, 1983. doi:10.1103/PhysRevLett.50.1395.

[4] M. Lemm, B. Nachtergaele, S. Warzel, and A. Young. Recursive spectral relations and the charge vs neutral gap in fractional quantum hall systems. *Journal of Mathematical Physics*, 67:011901, 2026. doi:10.1063/5.0292975; arXiv:2507.18814.

[5] B. Nachtergaele, S. Warzel, and A. Young. Spectral gaps and incompressibility in a $\nu = 1/3$ fractional quantum hall system. *Communications in Mathematical Physics*, 383:1093–1149, 2021. doi:10.1007/s00220-021-03997-0.

[6] G. Ortiz, Z. Nussinov, J. Dukelsky, and A. Seidel. Repulsive interactions in quantum hall systems as a pairing problem. *Physical Review B*, 88:165303, 2013. doi:10.1103/PhysRevB.88.165303.

[7] K. S. Rai, I. Kull, P. Emonts, J. Tura, N. Schuch, and F. Baccari. A hierarchy of spectral gap certificates for frustration-free spin systems. *Quantum*, 10:2065, 2026. doi:10.22331/q-2026-04-13-2065; arXiv:2411.03680.

[8] N. Rougerie. On the Laughlin function and its perturbations. *Séminaire Laurent Schwartz—EDP et applications*, pages 1–17, 2018–2019. Exposé II. doi:10.5802/slsedp.131; arXiv:1906.11656.

[9] R. Seiringer and J. Yngvason. Emergence of Haldane pseudo-potentials in systems with short-range interactions. *Journal of Statistical Physics*, 181:448–464, 2020. doi:10.1007/s10955-020-02586-0; arXiv:2001.07144.

[10] S. A. Trugman and S. Kivelson. Exact results for the fractional quantum hall effect with general interactions. *Physical Review B*, 31:5280–5284, 1985. doi:10.1103/PhysRevB.31.5280.

[11] S. Warzel and A. Young. The spectral gap of a fractional quantum hall system on a thin torus. *Journal of Mathematical Physics*, 63:041901, 2022. doi:10.1063/5.0084677; arXiv:2112.13764.

[12] Y. M. Zhao and A. Arima. Classification of states of single-j fermions with J-pairing interaction. *Physical Review C*, 70:034306, 2004. doi:10.1103/PhysRevC.70.034306; arXiv:nucl-th/0404042.